UNIQUENESS OF HYPERFINITE TYPE II_1 FACTOR

Abstract. These notes are devoted to the exposition of Murray & von Neumann's proof of the theorem that any two hyperfinite II_1 factors are isomorphic.

1. Von Neumann algebras, factors

Let $\mathscr{H}$ be a Hilbert space. Throughout these notes we assume that $\mathscr{H}$ is separable. A *von Neumann algebra* is a self-adjoint unital $*$-subalgebra M of $\mathrm{B}(\mathscr{H})$ closed in the w-topology.[a] Equivalently $M = M''$, where for any $S \subset \mathrm{B}(\mathscr{H})$ we define its *commutant* by

$$S' = \bigl\{x \in \mathrm{B}(\mathscr{H}) \bigm| xs = sx \text{ for all } s \in S\bigr\}$$

and *double commutant* $S'' = (S')'$.

For a convex set, being closed in the w-topology is equivalent to being closed in a number of other topologies like

- the so-topology defined by family of seminorms
$$\mathrm{B}(\mathscr{H}) \ni a \longmapsto \|a\xi\|, \qquad \xi \in \mathscr{H}$$
- the wo-topology defined by family of seminorms
$$\mathrm{B}(\mathscr{H}) \ni a \longmapsto \bigl|(\eta|a\xi)\bigr|, \qquad \xi, \eta \in \mathscr{H}.$$

A von Neumann algebra M is a *factor* if the *center* $\mathscr{Z}(M) = M \cap M'$ of M is equal to $\mathbb{C}\mathbb{1}$.

We denote the set of *projections* in M by

$$\mathscr{P}(M) = \bigl\{p \in M \bigm| p^*p = p\bigr\}$$

and *unitary elements* by

$$\mathscr{U}(M) = \bigl\{u \in M \bigm| u^*u = uu^* = \mathbb{1}\bigr\}.$$

By a subalgebra of a von Neumann algebra N we shall almost always[b] mean a unital w-closed $*$-subalgebra $N \subset M$. A *subfactor* is a subalgebra which is a factor.

1.1. Comparison theory, comparability.

Definition 1.1. Let $p, q \in \mathscr{P}(M)$. We say that

(1) $p \leq q$ if $pq = p$,
(2) $p \sim q$ if there is $u \in M$ such that $u^*u = p$ and $uu^* = q$,
(3) $p \preccurlyeq q$ if there is $r \in \mathscr{P}(M)$ such that $p \sim r \leq q$.

Both "$\leq$" is a partial order on $\mathscr{P}(M)$ and "$\preccurlyeq$" is a partial order on $\mathscr{P}(M)/\sim$. The antisymmetry of "$\preccurlyeq$" is expressed by the so called Schröder-Bernstein theorem.

Theorem 1.2. *For $p, q \in \mathscr{P}(M)$ there exists $z \in \mathscr{P}\bigl(\mathscr{Z}(M)\bigr)$ such that*

$$pz \preccurlyeq qz, \qquad p(\mathbb{1} - z) \succcurlyeq q(\mathbb{1} - z).$$

It follows that if M is a factor then "$\preccurlyeq$" defines a total order on $\mathscr{P}(M)$.

[a] The w-topology is the weak* topology coming from the duality $\mathrm{B}(\mathscr{H}) = \mathscr{T}(\mathscr{H})^*$.
[b] I.e. unless explicitly sated otherwise.

1.2. Type II_1 factors.

Definition 1.3. A factor M is of type II_1 if M is infinite-dimensional and M possesses a faithful tracial normal state.

The true definition of a type II_1 factor is that M is such if M is a factor without minimal projections and whose unit $\mathbb{1}$ is a *finite projection,* i.e. that $\mathbb{1}$ is not equivalent to a projection $p \lneqq \mathbb{1}$. It is a theorem that M is a II_1 factor if and only if it satisfies the conditions of Definition 1.3. It is also known that any trace on a factor is automatically faithful and normal (so these conditions are not necessary in Definition 1.3). One can also prove that a trace on a factor is unique (if it exists).

Theorem 1.4. *Let M be a II_1 factor and let τ be its trace. Then for any $r \in [0,1]$ there exists $p \in \mathscr{P}(M)$ such that $\tau(p) = r$.*

2. Preliminary results

Let M be a von Neumann algebra with a faithful state τ. Then we can define a positive definite scalar product on M by
$$(a|b) = \tau(a^*b).$$
The associated norm $a \mapsto \sqrt{(a|a)}$ will be denoted by $\|\cdot\|_2$ (the state τ will be fixed). We have that for any $a, b \in M$ the following estimate holds:
$$\|ab\|_2 \leq \|a\|\|b\|_2.$$
Indeed,
$$\|ab\|_2^2 = (ab|ab) = \tau\big((ab)^*ab\big) = \tau(b^*a^*ab) \leq \|a\|^2\tau(b^*b) = \|a\|^2\|b\|_2^2$$
because $a^*a \leq \|a\|^2\mathbb{1}$ and τ is positive.

It τ is a *trace* then we have also
$$\|bc\|_2 \leq \|b\|_2\|c\|$$
so that
$$\|abc\|_2 \leq \|a\|\|b\|_2\|c\|$$
for all $a, b, c \in M$.

From now on we assume that M is a von Neumann algebra with a faithful tracial state. The symbol $\|\cdot\|_2$ will denote the associated norm.

Lemma 2.1. *For any selfadjoint $a, b \in M$ we have*
$$\left\|\tfrac{a+\mathrm{i}\mathbb{1}}{a-\mathrm{i}\mathbb{1}} - \tfrac{b+\mathrm{i}\mathbb{1}}{b-\mathrm{i}\mathbb{1}}\right\|_2 \leq 2\|a-b\|_2.$$

Proof. We have
$$\begin{aligned}\tfrac{a+\mathrm{i}\mathbb{1}}{a-\mathrm{i}\mathbb{1}} - \tfrac{b+\mathrm{i}\mathbb{1}}{b-\mathrm{i}\mathbb{1}} &= (a-\mathrm{i}\mathbb{1})^{-1}\big((a+\mathrm{i}\mathbb{1})(b-\mathrm{i}\mathbb{1}) - (a-\mathrm{i}\mathbb{1})(b+\mathrm{i}\mathbb{1})\big)(b-\mathrm{i}\mathbb{1})^{-1}\\ &= -2\mathrm{i}(a-\mathrm{i}\mathbb{1})^{-1}(a-b)(b-\mathrm{i}\mathbb{1})^{-1}.\end{aligned}$$
Therefore, since $\|(a-\mathrm{i}\mathbb{1})^{-1}\|, \|(b-\mathrm{i}\mathbb{1})^{-1}\| \leq 1$ (as $a = a^*$ and $b = b^*$) we obtain
$$\left\|\tfrac{a+\mathrm{i}\mathbb{1}}{a-\mathrm{i}\mathbb{1}} - \tfrac{b+\mathrm{i}\mathbb{1}}{b-\mathrm{i}\mathbb{1}}\right\|_2 \leq 2\|(a-\mathrm{i}\mathbb{1})^{-1}\|\|a-b\|_2\|(b-\mathrm{i}\mathbb{1})^{-1}\| \leq 2\|a-b\|_2.$$
∎

Lemma 2.2. *Let $f \in \mathrm{C}(\mathbb{T})$. For any $\varepsilon > 0$ there exists $\omega_1(\varepsilon) > 0$ such that*
$$\left(\begin{array}{c} u, v \in \mathscr{U}(M),\\ \|u-v\|_2 < \omega_1(\varepsilon)\end{array}\right) \Longrightarrow \big(\big\|f(u) - f(v)\big\|_2 < \varepsilon\big).$$

Proof. If $f_1, f_2 \in \mathrm{C}(\mathbb{T})$ then[c]

$$\begin{aligned}\big\|f_2(u)-f_2(v)\big\|_2 &\le \big\|f_1(u)-f_2(u)\big\|_2+\big\|f_2(u)-f_2(v)\big\|_2+\big\|f_1(v)-f_2(v)\big\|_2\\ &\le \big\|f_2-f_2\big\|_\infty+\big\|f_2(u)-f_2(v)\big\|_2+\big\|f_2-f_2\big\|_\infty.\end{aligned}$$

Therefore, if for any ε a constant $\widetilde{\omega}_1(\varepsilon)$ satisfying

$$\begin{pmatrix} u, v \in \mathscr{U}(M), \\ \|u-v\|_2 < \widetilde{\omega}_1(\varepsilon)\end{pmatrix} \Longrightarrow \big(\big\|\widetilde{f}(u)-\widetilde{f}(v)\big\|_2 < \varepsilon\big).$$

can be found for $\widetilde{f}$ a trigonometric polynomial, then an analogous constant can be given for any $f \in \mathrm{C}(\mathbb{T})$ (indeed, take a trigonometric polynomial $\widetilde{f}$ such that $\|\widetilde{f}-f\|_\infty < \frac{\varepsilon}{3}$ and define $\omega_1(\varepsilon) = \widetilde{\omega}_1(\frac{\varepsilon}{3})$).

Therefore let us assume that f is a trigonometric polynomial

$$f(\mu) = \sum_{n=-N}^{M} \alpha_n \mu^n, \qquad \mu \in \mathbb{T}.$$

Then

$$\begin{aligned}\big\|f(u)-f(v)\big\|_2 &\le \sum_{n=-N}^{M} |\alpha_n| \|u^n - v^n\|_2\\ &= \sum_{n=1}^{N} |\alpha_{-n}| \|u^{*n} - v^{*n}\|_2 + \sum_{n=0}^{M} |\alpha_n| \|u^n - v^n\|_2\\ &= \sum_{n=1}^{N} |\alpha_{-n}| \|u^{n} - v^{n}\|_2 + \sum_{n=0}^{M} |\alpha_n| \|u^n - v^n\|_2\\ &= \sum_{n=1}^{N} |\alpha_{-n}| \Big\|\sum_{k=1}^{n} u^{k-1}(v-v)v^{k-1}\Big\|_2 + \sum_{n=0}^{M} |\alpha_n| \Big\|\sum_{k=1}^{n} u^{k-1}(v-v)v^{k-1}\Big\|_2\\ &\le \sum_{n=1}^{N} |\alpha_{-n}| \Big(\sum_{k=1}^{n} \|u^{k-1}(u-v)v^{k-1}\|_2\Big)\\ &\qquad + \sum_{n=0}^{M} |\alpha_n| \Big(\sum_{k=1}^{n} \|u^{k-1}(u-v)v^{k-1}\|_2\Big)\\ &\le \sum_{n=1}^{N} |\alpha_{-n}| \Big(\sum_{k=1}^{n} \|(u-v)\|_2\Big) + \sum_{n=0}^{M} |\alpha_n| \Big(\sum_{k=1}^{n} \|(u-v)\|_2\Big)\\ &= \|(u-v)\|_2 \sum_{n=-N}^{M} |n||\alpha_n|\end{aligned}$$

Therefore for such f we can take

$$\omega_1(\varepsilon) = \Big(\sum_{n=-N}^{M} |n||\alpha_n|\Big)^{-1} \varepsilon.$$

□

Lemma 2.3. *Let $g \in \mathrm{C}_0(\mathbb{R})^+$ for which $\lim_{t\to+\infty} g(t) = \lim_{t\to-\infty} g(t)$. Then for any $\varepsilon > 0$ there exists $\omega_2(\varepsilon) > 0$ such that*

$$\begin{pmatrix} a, b \in M,\ a = a^*,\ b = b^*, \\ \|a-b\|_2 < \omega_2(\varepsilon)\end{pmatrix} \Longrightarrow \big(\big\|g(a)-g(b)\big\|_2 < \varepsilon\big).$$

[c]Note that $\|a\|_2 = \sqrt{\tau(a^*a)} \le \sqrt{\|a^*a\|} = \|a\|$ for any $a \in M$.

Proof. A function g with the properties as in the statement of the lemma corresponds to a unique continuous function f on $\mathbb{T}$ via

$$f(\mu)=\begin{cases} g\big(\mathrm{i}\frac{\mu+1}{\mu-1}\big) & \mu\neq 1,\\ \lim\limits_{t\to\pm\infty} g(t) & \mu=1.\end{cases}$$

In other words

$$g(t)=f\big(\tfrac{t+\mathrm{i}}{t-\mathrm{i}}\big).$$

Let

$$u=\tfrac{a+\mathrm{i}\mathbb{1}}{a-\mathrm{i}\mathbb{1}},\qquad v=\tfrac{b+\mathrm{i}\mathbb{1}}{b-\mathrm{i}\mathbb{1}}.$$

Then

$$g(a)=f(u)\quad\text{and}\quad g(b)=f(v).$$

As $f\in \mathrm{C}(\mathbb{T})$, Lemma 2.2 gives us a function $\varepsilon\mapsto\omega_1(\varepsilon)$. Put $\omega_2(\varepsilon)=\frac{1}{2}\omega_1(\varepsilon)$. Then since

$$\|u-v\|_2\leq 2\|a-b\|_2$$

(by Lemma 2.1), we have that $\|a-b\|_2<\omega_2(\varepsilon)$ guarantees that

$$\big\|g(a)-g(b)\big\|_2=\big\|f(u)-f(v)\big\|_2<\varepsilon.$$

$\square$

Proposition 2.4. *For any $\varepsilon>0$ there exists $\omega_3(\varepsilon)>0$ such that if N is a subalgebra of M, $e\in\mathscr{P}(M)$ and $a\in N$ is such that $\|a-e\|_2<\omega_3(\varepsilon)$ then there exists $f\in\mathscr{P}(N)$ such that $\|f-e\|_2<\varepsilon$.*

Proof. We have

$$\big\|\tfrac12(a+a^*)-e\big\|_2=\big\|\tfrac12(a+e)-\tfrac12(a^*-e)\big\|_2\leq\tfrac12\big(\|a-e\|_2+\|a^*-e\|_2\big)=\|a-e\|_2$$

In other words we may (by replacing a by $\frac12(a+a^*)$ if necessary) that $a=a^*$ and any estimate on $\|a-e\|_2$ will still hold.

Define $g_1,g_2\in\mathrm{C}_0(\mathbb{R})^+$ by

$$g_1(t)=\begin{cases}1 & |t|\geq 1,\\ 2|t|-1 & \frac12\leq|t|<1,\\ 0 & |t|\leq\frac12,\end{cases}$$

$$g_2(t)=\begin{cases}1 & |t|\geq\frac12,\\ 2|t| & |t|<\frac12.\end{cases}$$

Then Lemma 2.3 applies to both g_1 and g_2. Let $\omega_2'(\varepsilon)$ be the smaller of the values $\omega_2(\varepsilon)$ which Lemma 2.3 gives for g_1 and g_2 and let

$$\omega_3(\varepsilon)=\omega_2'\big(\tfrac{\varepsilon}{2}\big).$$

Now let $h(t)=\chi\big(|t|\geq\frac12\big)$ and

$$f=h(a)\in\mathscr{P}(M).$$

Since $g_1(t)=g_2(t)=h(t)=t$ for any t in the spectrum of f

$$g_1(f)=g_2(f)=h(t)=f.$$

Moreover, since

$$\big(1-h(t)\big)g_1(t)+h(t)g_2(t)=h(t)^{\mathrm{d}}$$

we have

$$(\mathbb{1}-f)g_1(a)+fg_2(a)=f.$$

If $\|a-e\|_2<\omega_3(\varepsilon)$ then

$$\big\|g_1(a)-g_1(e)\big\|_2<\tfrac{\varepsilon}{2}\quad\text{and}\quad\big\|g_2(a)-g_2(e)\big\|_2<\tfrac{\varepsilon}{2}.$$

[d]In fact $\big(1-h(t)\big)g_1(t)=0$ and $h(t)g_2(t)=h(t)$.

But $g_1(e) = g_2(e) = e$ (both functions coincide with identity on spectrum of e) so that
$$\big\|g_1(a) - e\big\|_2 < \tfrac{\varepsilon}{2} \quad \text{and} \quad \big\|g_2(a) - e\big\|_2 < \tfrac{\varepsilon}{2}.$$
It follows that
$$\begin{aligned}\big\|(\mathbb{1} - f)\big(g_1(a) - e\big) + f\big(g_2(a) - e\big)\big\|_2 &\leq \big\|(\mathbb{1} - f)\big(g_1(a) - e\big)\big\|_2 + \big\|f\big(g_2(a) - e\big)\big\|_2 \\ &= \|\mathbb{1} - f\|\big\|g_1(a) - e\big\|_2 + \|f\|\big\|g_2(a) - e\big\|_2 \\ &\leq \big\|g_1(a) - e\big\|_2 + \big\|g_2(a) - e\big\|_2 < \varepsilon.\end{aligned}$$
Now we note that
$$(\mathbb{1} - f)\big(g_1(a) - e\big) + f\big(g_2(a) - e\big) = \big((\mathbb{1} - f)g_1(a) + fg_2(a)\big) - e = f - e$$
so that $\|f - e\|_2 < \varepsilon$. □

The next proposition can be proved using Lemma 2.4

Proposition 2.5. *Let $S \subset M$ be an algebra. Then the following conditions are equivalent:*

(1) *S is strongly closed,*
(2) *S is closed in the metric defined by $\|\cdot\|_2$.*

Lemma 2.6. *For any $\varepsilon > 0$ there exists $\omega_4(\varepsilon) > 0$ such that for any $e, f \in \mathscr{P}(M)$ such that*
$$\|e - f\|_2 < \omega_4(\varepsilon)$$
there exists a partial isometry $w \in M$ with $w^*w \leq e$ and $ww^* \leq f$ and such that
$$\|e - w\|_2 < \varepsilon.$$

Proof. Let $a = fe$ has polar decomposition $a = w|a|$ with $|a|, w \in M$. The partial isometry w satisfies
$$w^*w \leq e$$
because
$$\operatorname{Ran} w^*w = (\ker a)^\perp = \overline{\operatorname{Ran} a^*} = \overline{\operatorname{Ran} ef} \subset \operatorname{Ran} e.$$
In particular $we = ww^*we = w(w^*w\,e) = w(w^*w) = ww^*w = w$. Similarly
$$ww^* \leq f$$
because
$$\operatorname{Ran} ww^* = \overline{\operatorname{Ran} a} = \overline{\operatorname{Ran} fe} \subset \operatorname{Ran} f.$$

Let $g \in \mathrm{C}_0(\mathbb{R})^+$ be the function
$$g(t) = \begin{cases} \sqrt{|t|} & |t| \leq 1, \\ 1 & |t| > 1. \end{cases}$$
Then $g\big(|a|^2\big) = |a|$ or, in other words,
$$g(efe) = |a|.$$
Lemma 2.3 provides for $\varepsilon > 0$ the number $\omega_2(\varepsilon)$ related to the function g. Let
$$\omega_4(\varepsilon) = \min\big\{\omega_2\big(\tfrac{\varepsilon}{2}\big), \tfrac{\varepsilon}{2}\big\}.$$

This choice guarantees that
$$\|e - efe\|_2 = \big\|e(e - f)e\big\|_2 \leq \|e - f\|_2 < \omega_4(\varepsilon) \leq \omega_2\big(\tfrac{\varepsilon}{2}\big),$$
so that
$$\big\|e - |a|\big\|_2 = \big\|e - g(efe)\big\|_2 = \big\|g(e) - g(efe)\big\|_2 < \tfrac{\varepsilon}{2}$$
and
$$\big\|w - a\big\|_2 = \big\|w - w|a|\big\|_2 = \big\|w\big(e - |a|\big)\big\|_2 < \tfrac{\varepsilon}{2}.$$
This means that $\|w - fe\|_2 < \frac{\varepsilon}{2}$.

On the other hand
$$\|fe - e\|_2 = \big\|(f - e)e\big\|_2 \leq \|f - e\|_2 < \tfrac{\varepsilon}{2}.$$

It follows that

$$\|w-e\|_2 \le \|w-fe\|_2 + \|fe-e\|_2 < \varepsilon.$$

$\square$

Lemma 2.7. *Let $u \in \mathscr{U}(M)$ and $w \in M$ is a partial isometry such that $ux = wx$ for all $x \in \operatorname{Ran} w^*w$, then*

$$\|u-w\|_2 \le \sqrt{2\|w-\mathbb{1}\|_2}$$

Proof. Since u and w agree on the orthogonal complement of $\ker w$ which is the range of w^* we have

$$uw^* = ww^*$$

and so

$$wu^* = (ww^*)^* = ww^*.$$

Therefore

$$(u-w)(u-w)^* = uu^* - wu^* - uw^* + ww^* = \mathbb{1} - ww^*$$

and

$$\|u-w\|_2 = \sqrt{\tau\big((u-w)(u-w)^*\big)} = \sqrt{\tau(\mathbb{1}-ww^*)}.$$

Finally

$$\begin{aligned}
\tau(\mathbb{1}-ww^*) &= \tau\big(-(w-\mathbb{1})\mathbb{1}^* - w(w-\mathbb{1})^*\big) \\
&= \tau\big(-(w-\mathbb{1})\mathbb{1}^*\big) - \tau\big(w(w-\mathbb{1})^*\big) \\
&\le \big|\tau\big((w-\mathbb{1})\mathbb{1}^*\big)\big| + \big|\tau\big(w(w-\mathbb{1})^*\big)\big| \\
&\le \big|\tau\big(\mathbb{1}^*(w-\mathbb{1})\big)\big| + \big|\tau\big((w-\mathbb{1})^*w\big)\big| \\
&= \big|(\mathbb{1}|w-\mathbb{1})\big| + \big|(w-\mathbb{1}|w)\big| \\
&\le \|\mathbb{1}\|_2\|w-\mathbb{1}\|_2 + \|w-\mathbb{1}\|_2\|w\|_2 \le 2\|\mathbb{1}-w\|_2
\end{aligned}$$

because $\|w\|_2^2 = \tau(w^*w) \le \tau(\mathbb{1}) = 1$. $\square$

Proposition 2.8. *Let M be a II_1 factor. For any $\varepsilon > 0$ there exists $\omega_5(\varepsilon) > 0$ such that for any $e, f \in \mathscr{P}(M)$ with $\tau(e) = \tau(f)$ and such that*

$$\|e-f\| < \omega_5(\varepsilon)$$

there exists $U \in \mathscr{U}(M)$ such that

$$e = UfU^*$$

and

$$\|U-\mathbb{1}\|_2 < \varepsilon.$$

Proof. We choose $\varepsilon' > 0$ such that

$$2(\sqrt{\varepsilon'} + \varepsilon') < \varepsilon$$

and we put

$$\omega_5(\varepsilon) = \omega_4(\varepsilon').$$

By Lemma 2.6 there is a partial isometry w_0 such that

$$w_0^*w_0 \le e, \qquad w_0w_0^* \le f$$

and

$$\|w_0 - e\|_2 < \varepsilon'.$$

Now we note that

$$\tau(e - w_0^*w_0) = \tau(f - w_0w_0^*).$$

Since M is a II_1 factor, there is a partial isometry v_0 such that

$$v_0^*v_0 = e - w_0^*w_0, \qquad v_0v_0^* = f - w_0w_0^*.$$

Put

$$u_0 = w_0 + v_0.$$

Then u_0 is a partial isometry and

$$u_0^* u_0 = e, \qquad u_0 u_0^* = f.$$

We can repeat the above construction with $\mathbb{1} - e$ and $\mathbb{1} - f$ in place of e and f because

$$\big\|(\mathbb{1} - e) - (\mathbb{1} - f)\big\|_2 = \|e - f\|_2 < \omega_4(\varepsilon').$$

This gives us a partial isometry $w_1 \in M$ such that

$$w_1^* w_1 \leq \mathbb{1} - e, \qquad w_1 w_1^* \leq \mathbb{1} - f$$

and

$$\big\|w_1 - (\mathbb{1} - e)\big\|_2 < \varepsilon'.$$

Also we get v_1 as above and a partial isometry $u_1 = w_1 + v_1$ with

$$u_1^* u_1 = \mathbb{1} - e, \qquad u_1 u_1^* = \mathbb{1} - f.$$

Define now

$$W = w_0 + w_1, \qquad U = u_0 + u_1.$$

It is simple to see that U is unitary and W is a partial isometry. Moreover

$$f = UeU^*$$

$(UeU^* = (u_0 + u_1)e(u_0^* + u_1^*) = (u_0 + u_1)u_0^* u_0(u_0^* + u_1^*) = (u_0 + u_1)u_0^* = u_0 u_0^* = f)$.

Moreover U and W agree on $(\ker W)^{\perp}$. Therefore, by Lemma 2.7,

$$\|U - W\|_2 \leq \sqrt{2\|W - \mathbb{1}\|_2}.$$

Recall that

$$\|W - \mathbb{1}\|_2 = \|w_0 + w_1 - \mathbb{1}\|_2 = \big\|w_0 - e + w_1 - (\mathbb{1} - e)\big\|_2 \leq \|w_0 - e\|_2 + \big\|w_1 - (\mathbb{1} - e)\big\|_2 < 2\varepsilon'.$$

Thus

$$\|U - \mathbb{1}\|_2 \leq \|U - W\|_2 + \|W - \mathbb{1}\|_2 \leq \sqrt{2\|W - \mathbb{1}\|_2} + \|W - \mathbb{1}\|_2 < \sqrt{4\varepsilon'} + 2\varepsilon' < \varepsilon.$$

□

3. Approximate finite-dimensionality

Definition 3.1. Let $(p_1, p_2, p_3, \dots)$ be a sequence of natural numbers. We say that M is AFD$(p_1, p_2, p_3, \dots)$ if there exists a sequence $(N_i)_{i \in \mathbb{N}}$ of subalgebras of M such that

(1) N_i is a factor of type I_{p_i},
(2) $N_1 \subset N_2 \subset N_3 \subset \cdots$,
(3) $M = \left(\bigcup\limits_{i=1}^{\infty} N_i \right)''$.

Remark 3.2. Two II_1 factors which are AFD$(p_1, p_2, \dots)$ for some sequence $(p_i)_{i \in \mathbb{N}}$ are isomorphic. Indeed, first we produce an isomorphism on dense subalgebras, both isomorphic to

$$\bigcup_{i=1}^{\infty} M_{p_i}.$$

To extend this isomorphism to completions we note that this isomorphism must preserve the trace which makes it isometric in $\|\cdot\|_2$. Hence the extension is so-continuous on bounded sets and thus normal.

Definition 3.3. We say that M is AFD(A) if

(1) for any $a_1, \dots, a_m \in M$,
(2) for any $\varepsilon > 0$

there exists $n \in \mathbb{N}$ such that for any $q \geq n$ there exists a subfactor $N \subset M$ such that

(1) N is a factor of type I_q,
(2) there are $b_1, \dots, b_m \in N$ such that

$$\|b_i - a_i\|_2 < \varepsilon, \qquad i = 1, \dots m.$$

Definition 3.4. We say that M is AFD(B) if

(1) for any $a_1, \dots, a_m \in M$,
(2) for any $\varepsilon > 0$

there exists a subalgebra $N \subset M$ such that

(1) $\dim N < \infty$,
(2) there are $b_1, \dots, b_m \in N$ such that

$$\|b_i - a_i\|_2 < \varepsilon, \qquad i = 1, \dots m.$$

Definition 3.5. We say that M is AFD(C) if there exists a sequence $(N_i)_{i\in\mathbb{N}}$ of subalgebras of M such that

(1) $\dim N_i < \infty$ for all i,
(2) $N_1 \subset N_2 \subset N_3 \subset \cdots$,
(3) $M = \left(\bigcup_{i=1}^{\infty} N_i\right)''$.

Remark 3.6. The last definition is applicable to any von Neumann algebra M (not necessarily a II_1 factor). An algebra which is generated by an increasing sum of finite-dimensional subalgebras is called *hyperfinite* or *injective.*[e]

Theorem 3.7 (Murray & von Neumann 1943)**.** *Let M be a II_1 factor on a separable Hilbert space and let $(p_1, p_2, p_3, \dots)$ be a sequence of natural numbers such that*

- $p_i \mid p_{i+1}$ *for all* i,
- $p_1 \xrightarrow[i\to\infty]{} \infty$.

Then the following conditions are equivalent:

(1) M *is* AFD$(p_1, p_2, p_3, \dots)$,
(2) M *is* AFD(A),
(3) M *is* AFD(B),
(4) M *is* AFD(C).

The obvious implications are:

$$\mathrm{AFD}(p_1, p_2, p_3, \dots) \quad\Longrightarrow\quad \mathrm{AFD(C)}$$

and

$$\mathrm{AFD(A)} \quad\Longrightarrow\quad \mathrm{AFD(B)}.$$

Also the implication

$$\mathrm{AFD(C)} \quad\Longrightarrow\quad \mathrm{AFD(B)}.$$

is not difficult due to Proposition 2.5.

4. Implication "AFD(A) ⇒ AFD$(p_1, p_2, p_3, \dots)$"

In this section we assume that M is a II_1 factor which is AFD(A).

Lemma 4.1. *For any*

- $p \in \mathbb{N}$,
- $a_1, \dots, a_m \in M$,
- $e \in \mathscr{P}(M)$ *such that* $\tau(e) = \frac{1}{p}$,
- $\varepsilon > 0$

there exists $n \in \mathbb{N}$ such that for any $q \geq n$ with $p \mid q$ there exists $N \subset M$ such that

(1) *N is a subfactor of type I_q,*
(2) *there exist $b_1, \dots, b_m \in N$ such that $\|b_i - a_i\|_2 < \varepsilon$,*
(3) *there exists $f \in \mathscr{P}(N)$ such that $\tau(f) = \frac{1}{p}$ and $\|f - e\|_2 < \varepsilon$.*

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[e]Such algebras happen to be precisely the injective objects of the category of von Neumann algebras with separable preduals and normal completely positive unital maps as morphisms.

Proof. Choose ε'' such that $\sqrt{\varepsilon''} + \varepsilon'' < \varepsilon$ and let $\varepsilon' = \min\{\varepsilon, \omega_3(\varepsilon'')\}$, where ω_3 is taken from Proposition 2.4. Then let us use the property AFD(A) for the data $a_1, \ldots, a_m, a_{m+1} = e$ and ε'.

This produces a natural n and for any $q \geq n$ a subfactor $N \subset M$ of type I_q and $b_1, \ldots, b_m, b_{m+1} \in N$ with $\|b_1 - a_i\|_2 < \varepsilon'$ for $i = 1, \ldots, m+1$.

By Proposition 2.4 there exists $f_1 \in \mathscr{P}(N)$ with $\|f_1 - e\|_2 < \varepsilon''$. We have

$$\big|\tau(f_1) - \tau(e)\big| = \big|\tau(f_1 - e)\big| \leq \|f_1 - e\|_2 < \varepsilon'' \tag{1}$$

(here we use the Schwartz inequality

$$\big|\tau(a^*b)\big| \leq \|a\|_2\|b\|_2$$

for $a = \mathbb{1}$ and $b = f_1 - e$).

Now N is of type I_q and we only take q such that $p \mid q$. Therefore we have $f \in \mathscr{P}(N)$ such that $\tau(f) = \frac{1}{p}$ and $f \leq f_1$. Now since $f_1 - f$ is a projection, we have

$$\|f_1 - f\|_2^2 = \tau\big((f_1 - f)^*(f_1 - f)\big) = \tau(f_1 - f) = \tau(f_1) - \tau(f).$$

Thus

$$\|f_1 - f\|_2 = \sqrt{\tau(f_1) - \tau(f)} = \sqrt{\big|\tau(f_1) - \tau(e)\big|} < \sqrt{\varepsilon''}$$

by (1).

It follows that

$$\|f - e\|_2 \leq \|f - f_1\|_2 + \|f_1 - e\|_2 < \sqrt{\varepsilon''} + \varepsilon'' < \varepsilon.$$

□

Lemma 4.2. *For any*

- $p \in \mathbb{N}$,
- $a_1, \ldots, a_m \in M$,
- $e \in \mathscr{P}(M)$ *such that* $\tau(e) = \frac{1}{p}$,
- $\varepsilon > 0$

there exists $n \in \mathbb{N}$ *such that for any* $q \geq n$ *with* $p \mid q$ *there exists* $N \subset M$ *such that*

1. N *is a subfactor of type* I_q,
2. *there exist* $b_1, \ldots, b_m \in N$ *such that* $\|b_i - a_i\|_2 < \varepsilon$,
3. $e \in N$.

Proof. The elements $a_1, \ldots, a_m$ and $\varepsilon > 0$ are given, so let

$$\varepsilon'' = \tfrac{\varepsilon}{2\max\limits_i\{\|a_i\|\}+1}$$

Then let $\varepsilon' = \min\{\omega_5(\varepsilon''), \varepsilon''\}$, where ω_5 comes from Proposition 2.8. Then let us apply Lemma 4.1 with p, $a_1, \ldots, a_m$, e, ε'. This gives us $n \in \mathbb{N}$ and for any $q \geq n$ with $p \mid q$ we obtain a subfactor $N_1 \subset M$ of type I_q with elements $b_1^1, \ldots, b_m^1 \in N_1$ and a projection $f \in N_1$ such that

$$\|b_i^1 - a_i\|_2 < \varepsilon', \qquad i = 1, \ldots, m$$

and $\|f - e\|_2 < \varepsilon'$.

From Proposition 2.8 we know that there exists $u \in \mathscr{U}(M)$ such that

$$e = ufu^* \qquad \text{and} \qquad \|u - \mathbb{1}\|_2 < \varepsilon''.$$

Define $N = uN_1u^*$. Then N is a subfactor of M of type I_q. Let $b_i = ub_i^1u^*$ $(i = 1, \ldots, m)$. Then $b_i \in N$ for all i. We have

$$\begin{aligned}
\|b_i - a_i\|_2 &= \|ub_i^1u^* - a_i\|_2 \\
&\leq \|ub_i^1u^* - ua_iu^*\|_2 + \|ua_ia^* - a_i\|_2 \\
&= \big\|u(b_i^1 - a_i)u^*\big\|_2 + \|ua_ia^* - a_i\|_2 \\
&\leq \|b_i^1 - a_i\|_2 + \|ua_ia^* - a_i\|_2 < \varepsilon' + \|ua_ia^* - a_i\|_2 \\
&\leq \varepsilon'' + \|ua_ia^* - a_i\|_2.
\end{aligned}$$

Moreover

$$\begin{aligned}\|ua_iu^* - a_i\|_2 &\le \|ua_iu^* - ua_i\|_2 + \|ua_i - a_i\|_2 \\ &= \big\|ua_i(u^* - \mathbb{1})\big\|_2 + \big\|(u - \mathbb{1})a_i\big\|_2 \\ &\le \|a_i\|\|u^* - \mathbb{1}\|_2 + \|u - \mathbb{1}\|_2\|a_i\| \\ &= 2\|a_i\|\|u - \mathbb{1}\|_2 < 2\|a_i\|\varepsilon' \le 2\|a_i\|\varepsilon''.\end{aligned}$$

Therefore $\|b_i - a_i\| < \big(2\|a_i\| + 1\big)\varepsilon'' < \varepsilon$ for all i. □

Lemma 4.3. *For any*

- $p \in \mathbb{N}$,
- $a_1, \dots, a_m \in M$,
- $e \in \mathscr{P}(M)$ *such that* $\tau(e) = \frac{1}{p}$ *and* $ea_i = a_ie = a_i$ *for* $i = 1, \dots, m$,
- $\varepsilon > 0$

there exists $n \in \mathbb{N}$ *such that for any* $q \ge n$ *with* $p \mid q$ *there exists* $N \subset M$ *such that*

(1) N *is a subfactor of type* I_q,
(2) $e \in N$,
(3) *there exist* $b_1, \dots, b_m \in N$ *such that* $\|b_i - a_i\|_2 < \varepsilon$ *and* $eb_i = b_ie = b_i$ *for* $i = 1, \dots, m$.

Proof. We use Lemma 4.2 for p, $a_1, \dots, a_m$, e, ε to get $n \in \mathbb{N}$ and for any $q \ge n$ with $p \mid q$ a subfactor $N \subset M$ of type I_q with elements $b_1^1, \dots, b_m^1$ such that

$$\|b_i^1 - a_i\|_2 < \varepsilon, \qquad (i = 1, \dots, m).$$

Define $b_i = eb_i^1e$ to get elements of N such that $b_ie = eb_i = b_i$. Moreover

$$\|b_i - a_i\|_2 = \big\|e(b_i^1 - a_i)e\big\|_2 < \varepsilon$$

for $i = 1, \dots, m$. □

Lemma 4.4. *For any*

- $p \in \mathbb{N}$,
- $a_1, \dots, a_m \in M$,
- $N_0 \subset M$ *a subfactor of type* I_p,
- $e \in \mathscr{P}(N_0)$ *such that* $\tau(e) = \frac{1}{p}$ *and* $ea_i = a_ie = a_i$ *for* $i = 1, \dots, m$,
- $\varepsilon > 0$

there exists $n \in \mathbb{N}$ *such that for any* $q \ge n$ *with* $p \mid q$ *there exists* $N \subset M$ *such that*

(1) N *is a subfactor of type* I_q,
(2) $N_0 \subset N$,
(3) *there exist* $b_1, \dots, b_m \in N$ *such that* $\|b_i - a_i\|_2 < \varepsilon$ *and* $eb_i = b_ie = b_i$ *for* $i = 1, \dots, m$.

Proof. We have $\tau\big|_{N_0} = \frac{1}{p}\,\mathrm{Tr}$. Also $e \in N_0$ a projection of Tr-trace equal to 1. Let $\{w_{k,l}\}_{k,l=1,\dots,p}$ be a matrix unit basis of N_0 such that $w_{1,1} = e$. We have

$$\sum_{k=1}^{p} w_{k,k} = \mathbb{1}.$$

From Lemma 4.3 applied to p, $a_1, \dots, a_m$, e and ε we obtain $n \in \mathbb{N}$ and for any $q \ge n$ with $p \mid q$ a subfactor $\widetilde{N} \subset M$ of type I_q and $b_1, \dots, b_m \in \widetilde{N}$ such that

$$\|b_i - a_i\|_2 < \varepsilon$$

and $eb_i = b_ie = b_i$ for all i.

Since $\widetilde{N}$ is a factor of type I_q and $p \mid q$ there is a system of matrix units $\{u_{k,l}\}_{k,l=1,\dots,p}$ in $\widetilde{N}$ such that

$$\sum_{l=1}^{p} u_{l,l} = \mathbb{1}$$

and $e = u_{1,1}$.

Define

$$U = \sum_{l=1}^{p} w_{l,1}u_{1,l} \in N_0\widetilde{N} \subset M.$$

Then we check that U is unitary:

$$\begin{aligned}
U^*U &= \left(\sum_{l=1}^{p} w_{l,1}u_{1,l}\right)^* \left(\sum_{k=1}^{p} w_{k,1}u_{1,k}\right) = \sum_{k,l=1}^{p} u_{l,1}w_{1,l}w_{k,1}u_{1,k} \\
&= \sum_{l=1}^{p} u_{1,l}\left(\sum_{k=1}^{p} w_{1,l}w_{k,1}u_{1,k}\right) = \sum_{l=1}^{p} u_{l,1}\left(\sum_{k=1}^{p} \delta_{l,k}w_{1,1}u_{1,k}\right) \\
&= \sum_{l=1}^{p} u_{l,1}w_{1,1}u_{1,l} = \sum_{l=1}^{p} u_{l,1}eu_{1,l} \\
&= \sum_{l=1}^{p} u_{l,1}u_{1,1}u_{1,l} = \sum_{l=1}^{p} u_{l,1}u_{1,l} = \sum_{l=1}^{p} u_{l,l} = \mathbb{1}.
\end{aligned}$$

and

$$\begin{aligned}
UU^* &= \left(\sum_{l=1}^{p} w_{l,1}u_{1,l}\right)\left(\sum_{k=1}^{p} w_{k,1}u1,k\right)^* \\
&= \sum_{k,l=1}^{p} w_{l,1}u_{1,l}u_{k,1}w_{1,k} = \sum_{k,l=1}^{p} w_{l,1}\delta_{l,k}u_{1,1}w_{1,k} \\
&= \sum_{k=1}^{p} w_{l,k}u_{1,1}w_{1,k} = \sum_{k=1}^{p} w_{l,k}ew_{1,k} \\
&= \sum_{k=1}^{p} w_{l,k}w_{1,1}w_{1,k} = \sum_{k=1}^{p} w_{l,k}w_{1,k} = \mathbb{1}
\end{aligned}$$

Moreover we have

$$\begin{aligned}
Ue &= \sum_{l=1}^{p} w_{l,1}u_{1,l}e = \sum_{l=1}^{p} w_{l,1}u_{1,l}u_{1,1} = \sum_{l=1}^{p} w_{l,1}\delta_{1,l}u_{1,1} = w_{1,1}u_{1,1} = e, \\
eU &= \sum_{l=1}^{p} ew_{l,1}u_{1,l} = \sum_{l=1}^{p} w_{1,1}w_{l,1}u_{1,l} = \sum_{l=1}^{p} \delta_{1,l}w_{l,1}u_{1,l} = w_{1,1}u_{1,1} = e.
\end{aligned}$$

and

$$\begin{aligned}
Uu_{k,l} &= \sum_{r=1}^{p} w_{r,1}u_{1,r}u_{k,l} = \sum_{r=1}^{p} w_{r,1}\delta r,ku_{1,l} = w_{k,1}u_{1,l}, \\
w_{k,l}U &= \sum_{r=1}^{p} w_{k,l}w_{r,1}u_{1,r} = \sum_{r=1}^{p} \delta l,rw_{k,1}u_{1,r} = w_{k,1}u_{1,l},
\end{aligned}$$

so that

$$Uu_{k,l}U^* = w_{k,l}, \qquad k,l = 1,\dots,p.$$

Now we define $N = U\widetilde{N}U^*$. Then obviously $N \subset M$ is a subfactor of type I_q. We have

- $b_i \in N$ for all i because $Ub_iU^* = U(eb_ie)U^* = (Ue)b_i(Ue)^* = eb_ie = b_i$ (remember that we had $\|b_i - a_i\|_2 < \varepsilon$ from the beginning).
- $N_0 \subset N$ because $w_{k,l} = Uu_{k,l}U^* \in N$ for all k,l and $N_0 = \mathrm{span}\{w_{k,l}\,|\,k,l = 1,\dots,p\}$.

□

Lemma 4.5. *For any*

- $p \in \mathbb{N}$,
- $a_1,\dots,a_m \in M$,

- *$N_0 \subset M$ a subfactor of type I_p,*
- *$\varepsilon > 0$*

there exists $n \in \mathbb{N}$ such that for any $q \geq n$ with $p \mid q$ there exists $N \subset M$ such that

(1) *N is a subfactor of type I_q,*
(2) *$N_0 \subset N$,*
(3) *there exist $b_1, \ldots, b_m \in N$ such that $\|b_i - a_i\|_2 < \varepsilon$.*

Proof. Let $\{w_{k,l}\}_{k=1,\ldots,p}$ be a matrix unit basis in N_0 and let $e = w_{1,1}$. Define for $i = 1, \ldots, m$ and $k, l = 1, \ldots, p$ elements

$$a^i_{k,l} = w_{1,k} a_i w_{l,1} \in M.$$

Note that we have

$$e a^i_{k,l} = a^i_{k,l} e = a^i_{k,l}$$

for $i = 1, \ldots, m$ and $k, l = 1, \ldots, p$.

Now we use Lemma 4.4 for p, $\{a^i_{k,l}\}_{\substack{i=1\ldots,m \\ k,l=1,\ldots,p}}$, e, N_0 and

$$\varepsilon' = \tfrac{\varepsilon}{p^2}.$$

We obtain $n \in \mathbb{N}$ and for any $q \geq n$ with $p \mid q$ a subfactor $N \subset M$ of type I_q such that $N_0 \subset N$ containing elements $\{b^i_{k,l}\}_{\substack{i=1\ldots,m \\ k,l=1,\ldots,p}}$ such that

$$\|b^i_{k,l} - a^i_{k,l}\|_2 < \varepsilon'$$

and[f]

$$e b^i_{k,l} = b^i_{k,l} e = b^i_{k,l}$$

for all i, k, l.

Define

$$b^i = \sum_{k,l=1}^{p} w_{k,1} b^i_{k,l} w_{1,l} \in N$$

(since $N_0 \subset N$). Now note that we have

$$a_i = \sum_{k,l=1}^{p} w_{k,1} a^i_{k,l} w_{1,l}$$

(indeed:

$$\sum_{k,l=1}^{p} w_{k,1} a^i_{k,l} w_{1,l} = \sum_{k,l=1}^{p} w_{k,1} w_{1,k} a_i w_{l,1} w_{1,l} = \sum_{k,l=1}^{p} w_{k,k} a_i w_{l,l} = \mathbb{1} a_i \mathbb{1} = a_i$$

for all i, k, l) so that

$$\|b_i - a_i\|_2 = \left\| \sum_{k,l=1}^{p} w_{k,1}(b^i_{k,l} - a^i_{k,l}) w_{1,l} \right\|_2 \leq \sum_{k,l=1}^{p} \|b^i_{k,l} - a^i_{k,l}\|_2 < p^2 \varepsilon' = \varepsilon.$$

□

Lemma 4.6. *Let $(p_1, p_2, p_3, \ldots)$ be a sequence of natural numbers such that*

- *$p_i \mid p_{i+1}$ for all i,*
- *$p_i \xrightarrow[i\to\infty]{} \infty$*

there exists a subsequence $(p_{i_k})_{k \in \mathbb{N}}$ of $(p_i)_{i \in \mathbb{N}}$ such that M is AFD$(p_{i_1}, p_{i_2}, p_{i_3}, \ldots)$.

[f] This last condition actually is not necessary for the rest of the proof.

Proof. First fix an so-dense[g] sequence $(a_i)_{i\in\mathbb{N}}$ in M. We will choose inductively indices $i_1.i_2, i_3, \dots$ and for each k a subfactor $N_k \subset M$ of type $\mathrm{I}_{p_{i_k}}$ in such a way that

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

and

$$M = \Big(\bigcup_{k=1}^{\infty} N_k\Big)''.$$

Let $p_{i_0} = 1$ and $N_0 = \mathbb{C}\mathbb{1}$. Now assume that $i_1, i_2, \dots, i_{k-1}$ and $N_1, N_2, \dots, N_{k-1}$ have been chosen so that N_l is a type $\mathrm{I}_{p_{i_l}}$ factor and $N_l \subset N_{l+1}$. We use Lemma 4.5 for $p_{i_{k-1}}$, $a_1, \dots, a_k$, N_{k-1} and $\varepsilon = \frac{1}{k}$. This gives a number $n \in \mathbb{N}$ and we choose i_k so that $p_{i_k} > \max\{n, p_{i_{k-1}}\}$. Then $p_{i_{k-1}} \mid p_{i_k}$. For this p_{i_k} there is a subfactor $N_k \subset N$ of type $\mathrm{I}_{p_{i_k}}$ such that $N_{k-1} \subset N_k$ and containing elements

$$b_1^k, \dots, b_k^k \in N_k$$

with

$$\|b_i^k - a_i\|_2 < \varepsilon = \tfrac{1}{k}$$

for $i = 1, \dots, k$. In particular, we have for $j \in \mathbb{N}$

$$\|b_j^k - a_j\|_2 \xrightarrow[k\to\infty]{} 0.$$

□

Fact 4.7. Let $r, q, s \in \mathbb{N}$ be such that $r \mid q$ and $q \mid s$. If R is a type I_r subfactor of a type I_s factor S then there exists a type I_q factor Q such that

$$R \subset Q \subset S.$$

Theorem 4.8. *Let $(p_1, p_2, p_3, \dots)$ be a sequence of natural numbers such that*

- $p_i \mid p_{i+1}$ *for all* i,
- $p_i \xrightarrow[i\to\infty]{} \infty$

Then M is $\mathrm{AFD}(p_1, p_2, p_3, \dots)$.

5. Implication "AFD(B) ⇒ AFD(A)"

If N is a finite-dimensional von Neumann algebra then N is a (finite) direct sum of type I factors:

$$N = \bigoplus_{s=1}^{r} M_{q_s}(\mathbb{C}).$$

We let $\{w_{k,l}^s\}_{\substack{s=1,\dots,r\\ k,l=1,\dots,q_s}}$ be the matrix units in N. Thus $\{w_{k,l}^s\}_{\substack{s=1,\dots,r\\ k,l=1,\dots,q_s}}$ is a basis of N. We will also use the symbols $E_1, \dots, E_r$ to denote the central projections:

$$E_s = \sum_{k=1}^{q_s} w_{k,k}^s.$$

Lemma 5.1. *Let M be a type II_1 factor. Then for any $q \in \mathbb{N}$ there exists a type I_q subfactor $N_0 \subset M$.*

Proof. There are projections $E_1, \dots, E_q$ in M such that

$$\sum_{i=1}^{q} E_i = \mathbb{1}$$

and $\tau(E_i) = \frac{1}{q}$ for all i. Indeed, take E_1 with $\tau(E_1) = \frac{1}{q}$ and then (by comparability) there exists E_2 with $\tau(E_2) = \frac{1}{q}$ and $E_2 \leq \mathbb{1} - E_1$ and so on.

Since $E_i \sim E_1$ for all i, there exists $w_{i,1} \in M$ such that

$$w_{i,1} w_{i,1}^* = E_i, \qquad w_{i,1}^* w_{i,1} = E_1.$$

[g]Here separability of $\mathscr{H}$ is needed.

Define
$$w_{i,j} = w_{i,1} w_{j,1}^*$$
for $i, j = 1, \ldots, q$ and let
$$N_0 = \operatorname{span}\{w_{i,j} \,|\, i, j = 1, \ldots, q\}.$$
□

Proposition 5.2. *Let $N \subset M$ be a not necessarily unital finite-dimensional subalgebra of M and let $q \in \mathbb{N}$. If*
$$q\tau(w_{1,1}^s) \in \mathbb{N}$$
for all $s \in \{1, \ldots, r\}$ then there exists a subfactor $\widetilde{N} \subset M$ of type I_q such that
$$N \subset \widetilde{N}.$$

Proof. For a fixed $s \in \{1, \ldots, r\}$ we have $w_{k,k}^s \sim w_{1,1}^s$ for all k because
$$w_{1,k}^s (w_{1,k}^s)^* = w_{1,1}^s, \qquad (w_{1,k}^s)^* w_{1,k}^s = w_{k,k}^s.$$
It follows that
$$\tau(w_{k,k}^s) = \tfrac{p_s}{q} \tag{2}$$
for some $p_s \in \mathbb{N}$.

We will assume that
$$\sum_{i=1}^{r} E_s = \mathbb{1}. \tag{3}$$
This is the case only if N is a unital subalgebra of M (i.e. what we always mean by a *subalgebra*), but here we explicitly allow the case when the unit of N is not equal to the one of M. The solution is to replace N by the algebra N^+ generated by N and $\mathbb{1}$. This means that we add to N one projection $E_{r+1} = \mathbb{1} - \sum\limits_{s=1}^{r} E_r$ and put $w_{1,1}^{r+1} = E_{r+1}$ (no other matrix units in this extra summand). We want to prove the proposition for this extended version N^+ of N and thus have it also for the N. The point is that we have to check that
$$q\tau(w_{1,1}^{r+1}) \in \mathbb{N}.$$
But this readily follows because
$$q\tau(w_{1,1}^{r+1}) = q - \sum_{s=1}^{r} q\tau(w_{1,1}^s).$$
We can either extend N to N^+ or, in other words, assume (3).

From (3) and (2) we have
$$\sum_{s=1}^{r} q_s p_s = q.$$

Define $l_0 = 0$ and
$$\begin{aligned} l_1 &= p_1 q_1, \\ l_1 &= p_1 q_1 + p_2 q_2, \\ &\ \vdots \\ l_r &= p_1 q_1 + \cdots + p_r q_r = q. \end{aligned}$$
Now recall from Lemma 5.1 that there exists a subfactor $N_0 \subset M$ of type I_q. Let $\{u_{i,j}\}_{i,j=1,\ldots,q}$ be its matrix unit basis. Define[h]
$$F_s = \sum_{k=l_{s-1}+1}^{l_s} u_{k,k}, \qquad s = 1, \ldots, r$$

[h]The projections $F_1, \ldots, F_r$ do not play any special role in the proof, but writing them out helps in seeing how the remaining matrix units are constructed.

and

$$W^s_{k,l} = \sum_{i=1}^{p_s} u_{l_{s-1}+(k-1)p_s+i,\, l_{s-1}+(l-1)p_s+i}, \qquad s = 1,\dots,r, \quad k,l = 1,\dots,q_s.$$

The projections $F_1,\dots,F_r$ are mutually orthogonal and sum up to $\mathbb{1}$, while for fixed s the elements $\{W^s_{k,l}\}_{k,l=1,\dots,q_s}$ are matrix units of order q_s with

$$\sum_{k=1}^{q_s} W^s_{k,k} = F_s.$$

Moreover $W^s_{k,l} \in N_0 \subset M$ for all s,k,l.

We have that $u_{i,i} \sim u_{1,1}$ for $i = 1,\dots,q$, and

$$\sum_{i=1}^{q} u_{i,i} = \mathbb{1},$$

so

$$\tau(u_{i,i}) = \tfrac{1}{q}$$

for all i. It follows that

$$\tau(W^s_{k,k}) = \tfrac{p_s}{q} = \tau(w^s_{k,k})$$

for all s and k. By comparability there exists $v_1,\dots,v_r \in M$ such that

$$v_s^* v_s = W^s_{1,1}, \qquad v_s v_s^* = w^s_{1,1}.$$

Put

$$U = \sum_{s=1}^{r}\sum_{k=1}^{q_s} w^s_{k,1} v_s W^s_{1,k} \in M.$$

Then

$$\begin{aligned}
U^*U &= \left(\sum_{s=1}^{r}\sum_{k=1}^{q_s} W^s_{k,1} v_s^* w^s_{1,k}\right)\left(\sum_{s'=1}^{r}\sum_{k'=1}^{q_{s'}} w^{s'}_{k',1} v_{s'} W^{s'}_{1,k'}\right) \\
&= \sum_{s,s',k,k'} W^s_{k,1} v_s^* \underbrace{w^s_{1,k} w^{s'}_{k',1}}_{\delta_{s,s'}\delta_{k,k'} w^s_{1,1}} v_{s'} W^{s'}_{1,k'} \\
&= \sum_{s,k} W^s_{k,1} v_s^* w^s_{1,1} v_s W^s_{1,k} \\
&= \sum_{s,k} W^s_{k,1} \underbrace{v_s^* v_s v_s^* v_s}_{(W^s_{1,1})^2} W^s_{1,k} \\
&= \sum_{s,k} W^s_{k,1} W^s_{1,1} W^s_{1,k} = \sum_{s,k} W^s_{k,1} W^s_{1,k} = \sum_{s,k} W^s_{k,k} = \mathbb{1}
\end{aligned}$$

and

$$\begin{aligned}
UU^* &= \left(\sum_{s=1}^{r}\sum_{k=1}^{q_s} w^s_{k,1} v_s W^s_{1,k}\right)\left(\sum_{s'=1}^{r}\sum_{k'=1}^{q_{s'}} W^{s'}_{k',1} v_{s'}^* w^{s'}_{1,k'}\right) \\
&= \sum_{s,s',k,k'} w^s_{k,1} v_s \underbrace{W^s_{1,k} W^{s'}_{k',1}}_{\delta_{s,s'}\delta_{k,k'} W^s_{1,1}} v_{s'}^* w^{s'}_{1,k'} \\
&= \sum_{s,k} w^s_{k,1} v_s W^s_{1,1} v_s^* w^s_{1,k} \\
&= \sum_{s,k} w^s_{k,1} \underbrace{v_s v_s^* v_s v_s^*}_{(w^s_{1,1})^2} w^s_{1,k} \\
&= \sum_{s,k} w^s_{k,1} w^s_{1,1} w^s_{1,k} = \sum_{s,k} w^s_{k,1} w^s_{1,k} = \sum_{s,k} w^s_{k,k} = \mathbb{1}
\end{aligned}$$

so that U is unitary. Moreover

$$UW^a_{b,c} = \sum_{s=1}^{r}\sum_{k=1}^{q_s} w^s_{k,1} v_s \underbrace{W^s_{1,k} W^a_{b,c}}_{\delta_{s,a}\delta_{k,b}W^s_{1,c}} = w^a_{b,1} v_a W^a_{1,b} W^a_{b,c} = w^a_{b,1} v_a W^a_{1,c},$$

$$w^a_{b,c} U = \sum_{s=1}^{r}\sum_{k=1}^{q_s} \underbrace{w^a_{b,c} w^s_{k,1}}_{\delta_{a,s}\delta_{c,k}w^a_{b,1}} v_s W^s_{1,k} = w^a_{b,1} v_a W^a_{1,c},$$

which means that

$$UW^s_{k,l} = w^s_{k,l} U$$

for all s, k, l. Now we define $\widetilde{N} = UN_0U^*$. Then $\widetilde{N}$ is a subfactor of M of type I_q and it contains the basis $\{w^s_{k,l}\}_{\substack{s=1,\dots,r\\ k,l=1,\dots,q_s}}$ of N, so $N \subset \widetilde{N}$. □

Remark 5.3. The proof of the above proposition may be also accomplished in a more direct way. Namely, one can construct $\widetilde{N}$ "around" N instead of taking an auxiliary I_q subfactor N_0, constructing matrix units inside N_0 corresponding to the basis of N and then a unitary which "rotates" N_0 onto a subfactor $\widetilde{N}$ containing N. In this way one would simply reprove Lemma 5.1 within the proof of Proposition 5.2. However, Lemma 5.1 illustrates a crucial feature of factors, so we included it in this presentation.

Lemma 5.4. *Let $N \subset M$ be a finite-dimensional subalgebra. Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any $q \geq n$ there is a projection $e \in \mathscr{P}(M \cap N')$ such that*

1. *$\tau(\mathbb{1} - e) < \varepsilon$,*
2. *$q\tau(ew^s_{1,1}) \in \mathbb{N}$ for $s = 1, \dots, r$.*

Proof. Let $n \in \mathbb{N}$ be the greater than the two numbers

$$\max_{1\leq s\leq r}\{\tau(w^s_{1,1})^{-1}\} \qquad \text{and} \qquad \tfrac{1}{\varepsilon}\sum_{t=1}^{r} q_t.$$

Then for $q \geq n$ we have

$$q\tau(w^s_{1,1}) \geq 1$$

for $s = 1, \dots, r$ and

$$q\varepsilon\left(\sum_{t=1}^{r} q_t\right)^{-1} \geq 1.$$

Therefore, for each s, there exists $k_s \in \mathbb{N}$ such that

$$q\left(\tau(w^s_{1,1}) - \varepsilon\left(\sum_{t=1}^{r} q_t\right)^{-1}\right) < k_s \leq q\tau(w^s_{1,1}).$$

In other words

$$\tau(w^s_{1,1}) - \varepsilon\left(\sum_{t=1}^{r} q_t\right)^{-1} < \tfrac{k_s}{q} \leq \tau(w^s_{1,1}).$$

and $\frac{k_s}{q} \neq 0$.

Now for each s there exists $g^s \in \mathscr{P}(M)$ such that

$$\tau(g^s) = \tfrac{k_s}{q} \qquad \text{and} \qquad g^s \leq w^s_{1,1}$$

(by comparability).

Of course we have

$$\tau(w^s_{1,1} - g^s) = \tau(w^s_{1,1}) - \tau(g^s) = \tau(w^s_{1,1}) - \tfrac{k_s}{q} < \varepsilon\left(\sum_{t=1}^{r} q_t\right)^{-1}.$$

For a fixed $s = 1, \dots, r$ and $k = 1, \dots, q_s$ we define $g_k^s = w_{k,1}^s g^s w_{1,k}^s \leq w_{k,k}^s$. Then $g_1^s = g^s, g_2^s, \dots, g_{q_s}^s$ are pairwise orthogonal and equivalent. Indeed: for $l \neq k$ we have

$$g_k^s \leq w_{k,k}^s \perp w_{l,l}^s \geq g_l^s$$

and

$$w_{k,l}^s g_l^s = w_{k,l}^s w_{l,1}^s g^s w_{1,l}^s = w_{k,1}^s g^s w_{1,l}^s = w_{k,1}^s g^s w_{1,k}^s w_{k,l}^s = g_k^s w_{k,l}^s. \tag{4}$$

Moreover if $s_1 \neq s_2$ or $l_1 \neq l_2$ then

$$w_{k,l_1}^{s_1} g_{l_2}^{s_2} = 0 \qquad \text{and} \qquad g_{l_2}^{s_2} w_{l_1,k}^{s_1} = 0.$$

Put

$$e = \sum_{s=1}^{r} \sum_{k=1}^{q_s} g_k^s \in M.$$

We have

$$\mathbb{1} - e = \sum_{s=1}^{r} \sum_{k=1}^{q_s} w_{k,k}^s - \sum_{s=1}^{r} \sum_{k=1}^{q_s} g_k^s = \sum_{s=1}^{r} \sum_{k=1}^{q_s} (w_{k,k}^s - g_k^s). \tag{5}$$

Now recall that

$$\tau(w_{1,1}^s - g_1^s) = \tau(w_{1,1}^s - g^s) < \varepsilon \left(\sum_{t=1}^{r} q_t \right)^{-1}.$$

Applying conjugation $x \mapsto w_{k,1}^s x w_{1,k}^s$ to the estimate above we obtain

$$\tau(w_{k,k}^s - g_k^s) = \tau(w_{1,1}^s - g^s) < \varepsilon \left(\sum_{t=1}^{r} q_t \right)^{-1},$$

and so, by (5),

$$\tau(\mathbb{1} - e) < \varepsilon.$$

Now the calculation

$$w_{k,l}^s e = \sum_{t=1}^{r} \sum_{p=1}^{q_t} w_{k,l}^s g_p^t = \sum_{t=1}^{r} \sum_{p=1}^{q_t} \delta_{s,t} \delta_{l,p} w_{k,l}^s g_p^t = w_{k,l}^s g_l^s,$$
$$e w_{k,l}^s = \sum_{t=1}^{r} \sum_{p=1}^{q_t} g_p^t w_{k,l}^s = \sum_{t=1}^{r} \sum_{p=1}^{q_t} \delta_{t,s} \delta_{p,k} g_p^t w_{k,l}^s = g_k^s w_{k,l}^s$$

and (4) imply that $e \in N'$.

Finally

$$e w_{1,1}^s = g_1^s = g^s,$$

so that

$$\tau(e w_{1,1}^s) = \tau(g^s) = \tfrac{k_s}{q}$$

and $q\tau(e w_{1,1}^s) = k_s \in \mathbb{N}$ for $s = 1, \dots, r$. ☐

Theorem 5.5. *Let M be a type* II$_1$ *factor which is* AFD(B). *Then M is* AFD(A).

Proof. Let $a_1, \dots, a_m$ and $\varepsilon > 0$ be given. We must show that there exists $n \in \mathbb{N}$ such that for any $q \geq n$ a subfactor $\widetilde{N} \subset M$ of type I$_q$ can be found containing elements $b_1, \dots, b_m$ such that

$$\|b_i - a_i\|_2 < \varepsilon.$$

Since M is AFD(B), there exists a finite-dimensional subalgebra $N \subset M$ and elements

$$b_1^0, \dots, b_m^0 \in N$$

such that

$$\|b_i^0 - a_i\|_2 < \tfrac{\varepsilon}{2}.$$

The algebra N has matrix unit basis $\{w_{k,l}^s\}_{\substack{s=1,\dots,r \\ k,l=1,\dots,q_s}}$. Let us now apply Lemma 5.4 to N and

$$\varepsilon' = \left(\frac{\varepsilon}{2 \max\limits_{1 \leq i \leq m} \|b_i^0\|} \right)^2.$$

Thus we obtain $n \in \mathbb{N}$ and for any $q \geq n$ a projection $e \in M \cap N'$ such that

$$\tau(\mathbb{1} - e) < \varepsilon'$$

and

$$q\tau(w_{1,1}^s) \in \mathbb{N}$$

for $s = 1, \dots, r$.

Consider now the finite-dimensional non-unital subalgebra $N_e = eNe = eN = Ne$. It has a matrix unit basis[i]

$$\left\{ew_{k,l}^s\right\}_{\substack{s=1,\dots,r \\ k,l=1,\dots,q_s}} \tag{6}$$

By Proposition 5.2 there exists a subfactor $\widetilde{N}$ of type I_q such that

$$N_e \subset \widetilde{N} \subset M.$$

Define now

$$b_i = eb_i^0 \in N_e \subset \widetilde{N}, \qquad i = 1, \cdots, m.$$

Note that since

$$\|\mathbb{1} - e\|_2 = \sqrt{\tau(\mathbb{1} - e)} < \sqrt{\varepsilon'},$$

we get

$$\|b_i^0 - b_i\|_2 = \left\|b_i^0(\mathbb{1} - e)\right\|_2 < \|b_i^0\|\sqrt{\varepsilon'}.$$

Thus by the definition of ε' we have

$$\|b_i - b_i^0\|_2 < \tfrac{\varepsilon}{2}$$

and we obtain

$$\|b_i - a_i\|_2 \leq \|b_i - b_i^0\|_2 + \|b_i^0 - a_i\|_2 < \varepsilon.$$

□

[i]To see that the set (6) is really a basis let us compute (using the fact that $e \in N'$ and that τ is a trace) the scalar product of its arbitrary elements:

$$\begin{aligned}
\tau\big((ew_{k,l}^s)^*(ew_{k',l'}^{s'})\big) &= \tau(w_{l,k}^s ew_{k',l'}^{s'}) \\
&= \tau(ew_{l,k}^s w_{k',l'}^{s'}) \\
&= \delta_{s,s'}\delta k, k'\tau(ew_{l,l'}^s) \\
&= \delta_{s,s'}\delta k, k'\tau(ew_{l,1}^s w_{1,1}^s w_{1,l'}^s) \\
&= \delta_{s,s'}\delta k, k'\tau(w_{l,1}^s ew_{1,1}^s w_{1,l'}^s) \\
&= \delta_{s,s'}\delta k, k'\tau(ew_{1,1}^s w_{1,l'}^s w_{l,1}^s) \\
&= \delta_{s,s'}\delta k, k'\delta_{l,l'}\tau(ew_{1,1}^s w_{1,1}^s) = \delta_{s,s'}\delta_{k,k'}\delta_{l,l'}\tau(ew_{1,1}^s).
\end{aligned}$$

It follows that the system (6) is orthogonal, hence linearly independent.